

Semisimple algebraic tensor categories

(Rainer Weissauer)

For a field k a monoidal k -linear category abelian category T is an abelian k -linear category with biadditive tensor functor $\otimes : T \times T \rightarrow T$, k -linear and exact in each variable, with associativity and commutativity constraints and unit element 1_T satisfying the axioms ACU of [SR]. Then T is called rigid, if every object X has a dual X^* with morphisms

$$\delta_X : 1_T \rightarrow X \otimes X^* \quad , \quad ev_X : X^* \otimes X \rightarrow 1_T$$

so that $(id_X \otimes ev_X) \circ (\delta_X \otimes id_X) = id_X$ and $(\delta_X \otimes id_{X^*}) \circ (id_{X^*} \otimes ev_X) = id_{X^*}$. See [CP] or [SR] for a detailed exposition.

Under the assumptions above, if T is a small category such that $End_T(1_T) \cong k$, the category T is called a ‘catégorie k -tensorielle’ in [D]. If in addition T is generated by one of its objects V as a tensor category, such that for some integer N the length $l_T(V^{\otimes r})$ in T is bounded by N^r for all r , the category T will be called an algebraic tensor category over k .

The typical example for an algebraic tensor category over k (see [D], p.228) is the category of finite dimensional k -linear ε -super representations

$$T = Rep_k(\mathbf{G}, \varepsilon)$$

of a super-affine groupscheme \mathbf{G} over k . The main result on algebraic tensor categories is the following

Theorem 1. ([D]) *Suppose k is algebraically closed of characteristic zero. Then any algebraic tensor category over k is of the form $Rep_k(\mathbf{G}, \varepsilon)$.*

So let k be algebraically closed of characteristic zero. Under this assumption it is then interesting to know the cases where the category $Rep_k(\mathbf{G}, \varepsilon)$ is a semisimple abelian category. It is very easy to see that this only depends on the super-affine groupscheme \mathbf{G} and not on the additional twist ε . In other words $Rep_k(\mathbf{G}, \varepsilon)$ is semisimple if and only if the category $Rep_k(\mathbf{G})$ of all k -linear finite dimensional super representations of \mathbf{G} is semisimple. More or less by definition $Rep_k(\mathbf{G})$ coincides with the tensor category $CoRep_k(A)$ of k -finite dimensional

A -comodules, where A is the super-affine Hopf algebra over k defined the coordinate ring $\mathcal{O}(\mathbf{G})$ of \mathbf{G} . If these categories are semisimple, we say \mathbf{G} is reductive.

For a super-affine groupscheme \mathbf{G} over a field k of characteristic zero k the reduced groupscheme of \mathbf{G} is an algebraic group G over k . The left-invariant super derivations of the underlying Hopf algebra A corresponding to \mathbf{G} define a finite dimensional Lie superalgebra $g = \text{Lie}(\mathbf{G})$ over k . A Lie superalgebra g over k will be called reductive if modulo its supercenter it is isomorphic to a direct sum of simple Lie superalgebras over k of the classical types A_n ($n \geq 1$), B_n ($n \geq 3$), C_n ($n \geq 2$), D_n ($n \geq 3$), E_6, E_7, E_8, G_2, F_4 and of the orthosymplectic simple supertypes BC_r ($r \geq 1$). We then show

Theorem 2. *\mathbf{G} is reductive if and only its reduced group G is a reductive algebraic group over k and its Lie superalgebra $\text{Lie}(\mathbf{G})$ is reductive over k .*

In particular \mathbf{G} is reductive if and only if its connected component \mathbf{G}^0 with respect to the Zariski topology is reductive. In the connected case we show that \mathbf{G} is reductive if and only if etale unramified coverings are connected.

For the proof of theorem 2 we pass from super-affine groupschemes \mathbf{G} over k defined by their super-affine Hopf coordinate algebra A over k , to their associated supergroups (G, g_-, Q) . Here G is the reduced group of \mathbf{G} . The even part g_+ of $\text{Lie}(\mathbf{G}) = g_+ \oplus g_-$ is the Lie algebra of G . The odd part g_- is an algebraic G -module, and the Lie superbracket defines a G -equivariant symmetric map $Q : g_- \times g_- \rightarrow g_+$. Together these data give rise to a triple (G, g_-, Q) called a supergroup or a Harish-Chandra triple. For a suitable notion of representations for supergroups then the following holds

Theorem 3. *The categories of k -finite dimensional super representations $\text{Rep}_k(\mathbf{G})$ and $\text{Rep}_k(G, g_-, Q)$ are equivalent as algebraic tensor categories over k .*

Theorem 3 allows us to reduce the proof of theorem 2 to the classical results on the reductivity of semisimple Lie superalgebras obtained by Djokovic and Hochschild [DH].

Affine super Hopf algebras

Let k be field of $\text{char}(k) \neq 2$ and A be a Hopf algebra with comultiplication, counit and antipode (m_A^*, e_A^*, i_A^*) over the field k . Suppose A is super-affine, i.e. suppose that as a ring A is a finitely generated super-commutative k -algebra such that (m_A^*, e_A^*, i_A^*) are morphisms in the category (salg) of super-commutative k -algebras.

Remark. The tensor product \otimes^ε of the category (salg) is the ordinary tensor product \otimes_k except that it carries an induced grading with additional sign rules for certain structures like the tensor product of super k -algebras etc. For a detailed exposition of this we refer to [DM].

For the $\mathbf{Z}/2\mathbf{Z}$ -grading $A = A_+ \oplus A_-$ defined by the super structure the super-commutativity rule $xy = (-1)^{|x||y|}yx$ implies $x^2 = 0$ for $x \in A_-$. Thus A_- and the ideal J generated by A_- in A are nilpotent. We call J the super radical of A . J is a Hopf ideal, i.e. $i_A^*(J) \subset J$, $e_A^*(J) = 0$ and

$$m_A^*(J) \subset J \otimes^\varepsilon A + A \otimes^\varepsilon J$$

as an immediate consequence of

$$J = A_- + (A_-)^2$$

and $m_A^*(A_-) \subset (A \otimes^\varepsilon A)_- \subset A_- \otimes^\varepsilon A + A \otimes^\varepsilon A_-$. Surjective Hopf algebra homomorphisms $\pi : A \rightarrow A'$ are in 1-1 correspondence with Hopf ideals $I = \text{Kern}(\pi)$ of A . Since A/J is even, the quotient

$$\pi : A \rightarrow B = A/J$$

defines an commutative affine Hopf algebra quotient B for which therefore

$$G = \text{Spec}(B)$$

is a group scheme of finite type over k . We say A is connected, if G is connected in the Zariski topology. Similar for the notion of being simply connected. If $\text{char}(k) = 0$, then G is automatically reduced by a result of Cartier. In this case the super radical J is the nilradical of A .

A-comodules

An A -comodule (V, Δ_V) is a k -super vector space V together with a k -superlinear map

$$\Delta_V : V \rightarrow V \otimes^\varepsilon A$$

satisfying the axioms (Modass) and (Modun) as in [S], p.30, i.e. the commutativity of

$$\begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes^\varepsilon A \\ \Delta_V \downarrow & & \downarrow \Delta_V \otimes^\varepsilon id_A \\ V \otimes^\varepsilon A & \xrightarrow{id_V \otimes^\varepsilon m_A^*} & V \otimes^\varepsilon A \otimes^\varepsilon A \end{array} \quad \begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes^\varepsilon A \\ id_V \downarrow & & \downarrow id_V \otimes^\varepsilon e_A^* \\ V & \xlongequal{\quad} & V \otimes^\varepsilon k \end{array}$$

The notion of A -comodule only depends on the cogeбра structure of A . With the obvious notion of A -comodule homomorphism (see [S], p.31) the category of A -comodules is an abelian category. Any A -comodule is a union of its k -finite dimensional A -submodules. The category $CoRep_k(A)$ of k -finite dimensional A -comodules is a k -linear rigid abelian (monoidal) tensor category (see [CP], p.141).

Example a). (A, m_A^*) itself is an A -comodule by the Hopf algebra axioms

$$\begin{array}{ccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ m_A^* \downarrow & & \downarrow m_A^* \otimes^\varepsilon id_A \\ A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon m_A^*} & (A \otimes^\varepsilon A) \otimes^\varepsilon A \end{array} \quad \begin{array}{ccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ id_A \downarrow & & \downarrow id_A \otimes^\varepsilon e_A^* \\ A & \xlongequal{\quad} & A \otimes^\varepsilon k \end{array}$$

Example b). For a k -super subvectorspace $V \subset A$ such that $m_A^*(V) \subset V \otimes^\varepsilon A$ the restriction $\Delta_V = m_A^*|_V$ defines an A -comodule (V, Δ_V) , a subcomodule of A .

Example c). Any Hopf algebra quotient $\pi : A \rightarrow B$ map makes A -comodules (V, Δ_V) into B -comodules (V, Δ) with respect to

$$\Delta = (id_V \otimes^\varepsilon \pi) \circ \Delta_V : V \rightarrow V \otimes^\varepsilon B.$$

This is a consequence of $(\pi \otimes^\varepsilon \pi) \circ m_A^* = m_B^* \circ \pi$ and $e_A^* \circ \pi = e_B^*$.

Representations

Suppose for an A -comodule (V, Δ_V) that the super vectorspace $V = k^{r|s}$ is finite dimensional with basis e_i for $i = 1 \dots r + s$. Then $\Delta_V(e_i) = \sum_j e_j \otimes^\varepsilon f_{ji}$ for certain $f_{ji} \in A$. The axiom (Modass) implies $m_A^*(f_{ki}) = \sum_k f_{kj} \otimes^\varepsilon f_{ji}$. Thus the coefficients f_{ji} define a homomorphism of super Hopf algebras

$$\mathcal{O}(Gl(V)) \rightarrow A$$

from the super Hopf algebra $A' = \mathcal{O}(Gl(V))$ of the general linear group of the super vector space V to A . Indeed as k -algebra $A' = k[X_{ij}, \det_1^{-1}, \det_2^{-1}]$ is generated by elements X_{kj} and the inverse of the determinants \det_1, \det_2 of the X_{ij} for $i, j \leq r$ resp. $i, j > r$ subject to the rule $m_{A'}^*(X_{ki}) = \sum_k X_{kj} \otimes^\varepsilon X_{ji}$. The elements X_{ij} are even iff $i, j \leq r$ or $i, j > r$. In other words, this defines a super representation of $Spec^\varepsilon(A)$, i.e. a homomorphism of super group schemes

$$Spec^\varepsilon(A) \rightarrow Gl(V) .$$

Conversely, it is easy to see that this defines a 1-1 correspondence between k -finite dimensional A -comodules V and finite k -linear dimensional super representations V of the Lie super group scheme $Spec^\varepsilon(A)$. The category $Rep_k(A)$ of such k -finite dimensional super representations of $Spec^\varepsilon(A)$ is an algebraic tensor category over k . The following is well known (see [D])

Lemma 1. *This correspondence induces a tensor-equivalence between the algebraic tensor categories $CoRep_k(A)$ and $Rep_k(A)$ over k .*

The functor of invariants $V \mapsto V^G$

For the k -groupscheme $G = Spec(B)$ consider the left-exact functor

$$V \mapsto V^G = Hom_{B-comod}(k, V)$$

from the category of B -comodules to the category of k -vectorspaces. The k -vectorspace $V^G \subseteq V$ can be identified with the maximal trivial B -subcomodule of V of all elements v in V for which

$$\Delta_M(v) = v \otimes 1_B .$$

We say a B -comodule V is free, if it is isomorphic to a B -comodule of the form $V = V_0 \otimes B = B^d$. Here V_0 is a k -vectorspace and $d = \dim_k(V_0)$. B -comodules will be called almost free, if they have a finite filtration by B -subcomodules whose successive quotients are free B -comodules. Notice $B^G = k \cdot 1_B$, since $v = (e_B^* \otimes id_B)(m_B^*(v)) = (e_B^* \otimes id_B)(\Delta_V(v)) = (e_B^*(v) \otimes 1_B) \in k \cdot 1_B$ for $v \in B^G$. Hence for free $V = V_0 \otimes B$

$$(V_0 \otimes B)^G = V_0.$$

Using bar-resolutions (see [DG], p.233ff) one can define derived functors $H^i(G, -)$ such that $H^0(G, V) = V^G$. In other words a short exact sequence of B -comodules gives rise to a long exact sequence of k -vectorspaces using the derived functors $H^i(G, -)$. By [DG], lemma 3.4

$$H^i(G, B) = 0 \quad , \quad i \geq 1$$

for any free B -comodule. Obviously $H^1(G, V) = 0$ for almost free B -comodules V . Hence

Lemma 2. *On the Grothendieck group of almost free B -comodules V*

$$rang_k(V) = \dim_k(V^G)$$

defines a homomorphism .

The Hopf ideals defined by J

Let A be a super-affine Hopf algebra over k . Then its super radical J is generated as an A -module by finitely many elements in A_- . If J is generated by s elements then it is easy to see that $J^{s+1} = 0$. Hence there exists a finite descending filtration by A -right (and left) ideals

$$0 \subset J^s \subset J^{s-1} \subset \dots \subset J^2 \subset J \subset A$$

whose successive quotients

$$V_i = J^i / J^{i+1}$$

are right (and left) $B = A/J$ -modules. Although the J^i are not B -modules a priori, they are B -subcomodules of the B -comodule (A, Δ) with structure map

$$\Delta = (id_A \otimes^\varepsilon \pi) \circ m_A^*$$

using the examples a), b) and c) above. There is a commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \\
 \uparrow & & & & \uparrow \\
 J^i & \xrightarrow{\Delta} & J^i \otimes^\varepsilon B & &
 \end{array}$$

since the image of $m_A^*(J^i) \subset m_A^*(J)^i \subset (J \otimes^\varepsilon A + A \otimes^\varepsilon J)^i \subset \sum_{a+b=i} J^a \otimes^\varepsilon J^b$ in $A \otimes^\varepsilon B$, under $id_A \otimes^\varepsilon \pi$, is contained in $J^i \otimes^\varepsilon B$. Thus J^i becomes a B -comodule. The V_i then are quotient B -comodules of the J^i in the obvious way.

Lemma 3. $V_i \cong B^{d_i}$ is finite free both as a B -right module and a B -comodule.

Proof. The k -linear structure map $\Delta : A \rightarrow A \otimes^\varepsilon B$ of the B -comodule A is A -linear in the following sense: For $a \in A$ and $x \in A$ of course $x \cdot a \in A$. Since π is A -linear

$$\Delta(x \cdot a) = (id_A \otimes^\varepsilon \pi)(m_A^*(x) \cdot m_A^*(a)) = \Delta(x) \bullet m_A^*(a)$$

where $A \otimes^\varepsilon B$ is viewed as a $A \otimes^\varepsilon A$ -right module in the obvious way. In other words $m_A^*(a) = \sum a_\nu \otimes a'_\nu$ acts on $y = a \otimes b$ via $y \bullet m_A^*(a) = \sum_\nu (-1)^{|a_\nu||b|} a \cdot a_\nu \otimes b \cdot a'_\nu$. Since $\Delta(J^i) \subset (J^i)$ the map Δ induces a quotient map

$$\Delta_V : V \rightarrow V \otimes^\varepsilon B$$

on $V = V_i = J^i/J^{i+1}$ making it to a B -comodule. The right action of A on V factors over the quotient ring B . Similarly the right action of $A \otimes^\varepsilon A$ on $V \otimes^\varepsilon B$ factors over the quotient ring $A/J \otimes^\varepsilon B = B \otimes^\varepsilon B$, so that now (*)

$$\Delta_V(x \cdot b) = \Delta_V(x) \bullet m_B^*(b)$$

is obvious: The composition of m_A^* with the projection $A \otimes^\varepsilon A \rightarrow A/J \otimes^\varepsilon B$ is equal to $m_B^* \circ \pi$.

It is the property (*) which makes the right B -module and right B -comodule V into a B -right *Hopf module* in the sense of [S], p.83. Since B is an ordinary Hopf algebra we can immediately apply [S], theorem 4.1.1. It states that

$$M \cong M^G \otimes B = B^d, \quad d = \dim_k(M^G)$$

as a Hopf right B -module and comodule for any Hopf right B -module and comodule M . Applied for $M = V$ we now use the fact that J , hence also V , are finitely generated B -right modules. Hence $d = d_i < \infty$ in our case. This proves our claim. QED

Therefore A is an almost free B -comodule. By lemma 2 this implies

Corollary 1. $\dim_k(A^G) = \sum_{i=0}^s d_i$ for $d_i = \text{rank}_B(J^i/J^{i+1})$.

Remark. We will see later in corollary 5 that for A affine super group scheme over k we have $d_i = \binom{s}{i}$. This will imply

$$\dim_k(A^G) = 2^s.$$

Lemma 4. A^G is a finite dimensional k -subalgebra of A .

Proof. $\Delta(v) = v \otimes 1_B$ and $\Delta(v') = v' \otimes 1_B$ imply $\Delta(v \cdot v') = \Delta(v) \cdot \Delta(v') = (v \otimes 1_B) \cdot (v' \otimes 1_B) = (v \cdot v') \otimes 1_B$. QED

Superderivations

Let (A, m_A^*, e_A^*, i_A^*) be an super-affine Hopf algebra over k . Let $m = m(A) = \text{kern}(e_A^*)$ be the maximal ideal of A at the identity. Target vectors $X \in (m/m^2)_\pm^*$ extend to even or odd k -linear superderivations $d_X : A \rightarrow k$ by composing $X : m/m^2 \rightarrow k$ with the projection $A = k \cdot 1 \oplus m \rightarrow m \rightarrow m/m^2$. Define k -superderivations

$$D_X : A \rightarrow A$$

by the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ & \searrow D_X & \downarrow id_A \otimes^\varepsilon d_X \\ & & A \end{array}$$

Then $d_X = e_A^* \circ D_X$ by definition. The k -superderivations $D_X : A \rightarrow A$ so constructed are left-invariant, i.e. for $D = D_X$ there exists a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ D \downarrow & & \downarrow id_A \otimes^\varepsilon D \\ A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \end{array}$$

by the coassociativity law $(id_A \otimes^\varepsilon m_A^*) \circ m_A^* = (m_A^* \otimes^\varepsilon id_A) \circ m_A^*$ and $A \otimes^\varepsilon (A \otimes^\varepsilon A) = (A \otimes^\varepsilon A) \otimes^\varepsilon A$. Indeed, if we apply $id_A \otimes^\varepsilon (id_A \otimes^\varepsilon d_X) = (id_A \otimes^\varepsilon id_A) \otimes^\varepsilon d_X$ on the left side of the coassociativity law, this becomes $(id_A \otimes^\varepsilon D_X) \circ m_A^*$. On the right side of the coassociativity law it becomes $m_A^* \circ D_X$.

Lemma 5. *There exists a canonical isomorphism $X \mapsto D_X$ of k -vector spaces*

$$(m/m^2)^* \rightarrow Lie(A)$$

between the tangent space at the identity element and the k -vector space $Lie(A)$ of all left-invariant k -superderivations of A .

Proof. The inverse map is $Lie(A) \ni D \mapsto d = e_A^* \circ D$. Since $d : A \rightarrow k$ is a k -superderivation, it must vanish on m^2 and on $k \cdot 1$. Hence $d = d_X$ for some $X \in (m/m^2)^*$. The left-invariant k -superderivation $D : A \rightarrow A$ is uniquely determined by its restriction $d = e_A^* \circ D$, since d determines D via the right vertical arrow $id_A \otimes^\varepsilon d$ of the composed commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ D \downarrow & & \downarrow id_A \otimes^\varepsilon D \\ A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\ & \searrow id_A & \downarrow id_A \otimes^\varepsilon e^* \\ & & A \end{array}$$

For any Hopf ideal J of A with quotient map $A \rightarrow B = A/J$ the cotangent space $m(A)/m(A)^2$ surjects onto the cotangent space $m(B)/m(B)^2$. Hence $Lie(B)$ injects into $Lie(A)$. QED

Lemma 6. *The image of the natural injection $Lie(B) \hookrightarrow Lie(A)$ is the space of left-invariant k -derivations D of A (as in the last lemma) with the property $D(J) \subset J$.*

Proof. Such D induce left-invariant derivations on the quotient $B = A/J$. So it suffices that $X \in Lie(B)$ implies $D_X(J) \subset J$. For this let $x : A \rightarrow B$ be the quotient map with kernel J , considered as a B -valued point of A . For $f \in A$ by definition $D_X(f)(x) = (x \otimes^\varepsilon d_X)(m_A^*(f))$. Now $(x \otimes^\varepsilon d_X)(m_A^*(f)) \in (x \otimes^\varepsilon d_X)(A \otimes^\varepsilon J + J \otimes^\varepsilon A)$ for $f \in J$ since J is a Hopf ideal. But $x(J) = 0$. On

the other hand $d_X(J) = 0$ for $X \in \text{Lie}(B)$, since $\text{Lie}(B)$ is the space of linear forms $d_X : m(A)/m(A)^2 \rightarrow k$ trivial on the image of J . Hence $D_X(f)(x) = 0$ or $D_X(f) \in J$. QED

The Lie algebra. The supercommutator $[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D$ of two k -super derivations D, D' is a k -superderivations. Since the super commutator of left-invariant derivations is left invariant, the finite dimensional k super-vector space

$$g = \text{Lie}(A) = (m(A)/m(A)^2)^*$$

defined by (A, m_A^*, e_A^*) becomes a Lie k -superalgebra with $g \cong k^{r|s}$ as a super vectorspace

$$g = g_+ \oplus g_- .$$

The super radical J . Notice $J = A_- + (A_-)^2$ implies $J^2 = (A_-)^2 + (A_-)^3$. Hence the quotient $J/J^2 = A_-/(A_-)^3$ is odd. Since J is nilpotent, we have $J \subset m(A)$ and $m(B) = m(A)/J$. Clearly the quotient $A_-/(A_-)^3 = J/J^2 \rightarrow J/(J \cap m(A)^2)$ again is odd. Since B is even, also $m(B)/m(B)^2$ is even with $g_+ = \text{Lie}(B) = (m(B)/m(B)^2)^*$ even. Hence the exact sequence

$$0 \rightarrow J/(m(A)^2 \cap J) \rightarrow m(A)/m(A)^2 \rightarrow m(B)/m(B)^2 \rightarrow 0$$

gives rise to a splitting of the super-vectorspace $\text{Lie}(A)$ with $\text{Lie}(G)$ even

$$\text{Lie}(A) = \text{Lie}(G) \oplus g_-$$

and with $g_- \cong (J/(J \cap m(A)^2))^*$ odd.

Fix a basis $\tilde{\theta}_i$ of $(V_1)^G = (J/J^2)^G$ and representatives $\theta_i \in J^G$ of the elements $\tilde{\theta}_i$. Then $J/J^2 = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot B$ as a B right-module. Consider the exact sequence of odd k -vectorspaces

$$0 \rightarrow K \rightarrow J/J^2 \rightarrow J/(J \cap m(A)^2) \rightarrow 0 .$$

We claim $K = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot m(B)$.

Since $\theta_i \in J \subset m(A)$, the right hand side is contained in K . Conversely elements $k \in K$ have odd representatives x in $J \cap m(A)^2$, or hence in $A_- \cap m(A)^2$. Notice $A_- \cap m(A)^2 = (A_- \cap m(A))(A_+ \cap m(A))$ by a case by case verification and the definition of the super graded ring structure on A . Since $m(A) \cap A_- \subset J$,

hence $A_- \cap m(A)^2 \subset J \cdot (A_+ \cap m(A))$. As $m(A)$ acts on J/J^2 via its quotient $m(B)$ therefore the image k of x is contained in $\bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot m(B)$. This proves the claim. As a consequence

$$(J/J^2)^G = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot k \cong (J/J^2)/K \cong J/(J \cap m(A)^2) \cong (g_-)^*.$$

Together with the lemma 6 this implies

Corollary 2. *The left-invariant derivations D_X for $X \in \text{Lie}(G) \subset \text{Lie}(A)$ respect the exact sequence defined by the super radical J*

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0.$$

A left-invariant superderivation $D_X \in \text{Lie}(A)$ preserves the super radical J if and only if $X \in \text{Lie}(G)$. Furthermore

$$\dim_k(g_-) = \text{rank}_B(J/J^2) = d_1.$$

Homomorphisms. A homomorphism $\Phi^* : A \rightarrow A'$ between super-affine k -Hopf algebras induces a map between the tangent spaces at the identity element, hence a k -linear map

$$\text{Lie}(\Phi) : \text{Lie}(A') \rightarrow \text{Lie}(A).$$

$\text{Lie}(\Phi)$ is a homomorphism of k -super Lie algebras, since $\Phi^* \circ D_X = D_{X'} \circ \Phi^*$ for $X' = \text{Lie}(\Phi)(X)$. [Reduce to $(\Phi^* \otimes^\varepsilon \Phi^*) \circ (id \otimes^\varepsilon d_X) = (id \otimes^\varepsilon d_{X'}) \circ \Phi^*$, hence to $\Phi^* \circ d_X = d_{X'}$.]

Adjoint action. The interior automorphism $\Phi^* = (\text{Int}_x)^*$ defined by a k -valued point of $\text{Spec}(A/I)$ induces a Lie algebra homomorphism $\text{Ad}(x) = \text{Lie}(\text{Int}_x)$ from $\text{Lie}(A)$ to $\text{Lie}(A)$. Obviously $\text{Ad}(x) \circ \text{Ad}(y) = \text{Ad}(xy)$. Hence $\text{Ad}(x)$ defines a k -linear representation on $\text{Lie}(A)$ of the underlying algebraic group G

$$\text{Ad} : G(k) \rightarrow \text{Gl}_k(\text{Lie}(A)).$$

This adjoint action respects the super structure, hence decomposes into representations Ad_\pm of G on g_+ and g_- respectively. Ad_+ is the usual adjoint action of $G(k)$ on its Lie algebra $g_+ = \text{Lie}(G)$.

Left versus right

Similar to left-invariant superderivations define right-invariant superderivations of a Hopf algebra A . The Lie superalgebra of the left-invariant and right-invariant superderivations are isomorphic (use the antipode). Left-invariant superderivations D and right-invariant superderivations D' of A supercommute. Use

$$(-1)^{|D||D'|} m_A^*(DD'x) = (D' \otimes^\varepsilon D)(m_A^*(x)) = m_A^*(D'Dx)$$

to show that their supercommutator $[D, D']$ is a derivation with $m_A^*([D, D'](x)) = 0$. Hence $[D, D'] = 0$ by applying the counit e_A^* .

Lemma 7. For quotients $B = A/I$ by a Hopf ideal I and $X \in \text{Lie}(B) \subset \text{Lie}(A)$ the left-invariant superderivations D_X of A preserve B -subcomodules V of A .

Proof. The commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \\ D_X \downarrow & & \downarrow id_A \otimes^\varepsilon D_X & & \downarrow id_A \otimes^\varepsilon D_X \\ V & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \end{array}$$

for D_X and $X \in \text{Lie}(B) \subset \text{Lie}(A)$ gives $\Delta \circ D_X = (id \otimes^\varepsilon D_X) \circ \Delta$ for the structure map $\Delta = (id_A \otimes^\varepsilon \pi) \circ m_A^*$ of the B -comodule A .

Next notice $(id_A \otimes^\varepsilon e_B^*) \circ \Delta = id_A$ and the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \\ \parallel & & \downarrow id_A \otimes^\varepsilon e_A^* & & \downarrow id_A \otimes^\varepsilon e_B^* \\ A & \xlongequal{\quad} & A & \xlongequal{\quad} & A \end{array}$$

Hence for $v \in V \subset A$ and $\Delta(v) = \sum_i v_i \otimes^\varepsilon b_i$ with $v_i \in V, b_i \in B$ the element

$$D_X(v) = (id_A \otimes^\varepsilon e_B^*) \circ \Delta(D_X(v))$$

is $(id_A \otimes^\varepsilon e_B^*)(\sum_i v_i \otimes^\varepsilon D_X(b_i)) = \sum_i v_i \cdot d_X(b_i)$ using left-equivariance of D_X as in first diagram above. Thus $D_X(v) \in V$ and $D_X(V) \subset V$. QED.

For $\Delta(v) = v \otimes^\varepsilon 1_B$ in particular $D_X(v) = 0$, since $d_X(1_B) = 0$.

Corollary 3. $D_X(A^G) = 0$ for all $D_X, X \in \text{Lie}(G)$ where $G = \text{Spec}^\varepsilon(A/I)$.

Corollary 4. A^G is stable under all right-invariant superderivations D'_X in $\text{Lie}(A)$.

Proof. This follows from the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \\ D'_X \downarrow & & \downarrow D'_X \otimes^\varepsilon id_A & & \downarrow id_A \otimes^\varepsilon D'_X \\ V & \xrightarrow{m_A^*} & A \otimes^\varepsilon A & \xrightarrow{id_A \otimes^\varepsilon \pi} & A \otimes^\varepsilon B \end{array}$$

which implies $\Delta \circ D'_X = (D'_X \otimes^\varepsilon id_B) \circ \Delta$ for the structure map Δ of the B -comodule A . For $v \in A^G$ by definition $\Delta(v) = v \otimes^\varepsilon 1_B$. Hence $\Delta(D'_X(v)) = (D'_X \otimes^\varepsilon id_B) \circ \Delta(v) = D'_X(v) \otimes^\varepsilon 1_B$. This shows $D'_X(A^G) \in A^G$. QED

The subring $A^G \subset A$

For the super radical J of A put $G = \text{spec}(B)$ and $B = A/J$ as before. Since (J/J^2) is odd and almost free, the quotient map $(J_-)^G \rightarrow (J/J^2)^G$ is surjective so that we can choose representatives $\theta_1, \dots, \theta_s \in (J_-)^G$ of a k -basis in $(J/J^2)^G$ so that the θ_i are also a B -basis of J/J^2 by lemma 2. Then by recursion modulo the J^n

$$J = \theta_1 \cdot A + \dots + \theta_s \cdot A.$$

The θ_i are odd. Hence by supercommutativity

$$\theta_i \theta_j = -\theta_j \theta_i.$$

For $I \subset \{1, \dots, s\}$ define $\theta_I = \theta_{i_1} \dots \theta_{i_n}$ if $I = \{i_1, \dots, i_n\}$ and $i_1 < \dots < i_n$. With these notations J^n is generated as an A -right module by the θ_I with $|I| = n$. Hence for the elements $\tilde{\theta}_I = \theta_I \bmod J^{n+1}$ in $(J^n/J^{n+1})^G$ we get

$$J^n/J^{n+1} = \sum_{|I|=n} \tilde{\theta}_I \cdot B.$$

We may replace by a B -right linear independent subset of T_n of the set of all the $\tilde{\theta}_I$, since we already know that J^n/J^{n+1} is a free B -right module generated by a k -basis of $(J^n/J^{n+1})^G$. Therefore

$$J^n/J^{n+1} = \bigoplus_{I \in T_n} \tilde{\theta}_I \cdot B$$

and

$$(J^n)^G / (J^{n+1})^G \cong (J^n / J^{n+1})^G \cong k^{\#T_n}.$$

Since $\theta_I \in A^G$, recursively now any element in A^G can be written as a superpolynomial in the elements $\theta_1, \dots, \theta_s$ by induction modulo the $A^G \cap J^n = (J^n)^G$. This defines a surjective k -algebra homomorphism $f : S^\varepsilon(k^{0|s}) \rightarrow A^G$ mapping the generators of the superpolynomial ring $S^\varepsilon(k^{0|s})$ to the θ_i .

Lemma 8. A^G is a superpolynomial ring $S^\varepsilon(k^{0|s})$ over k in the odd variables θ_i .

Proof. Recall $(J/J^2) \cong (g_-)^*$. This means that we can find s odd right-invariant superderivations D'_i in $g_- \subset \text{Lie}(A)$ such that $e_A^*(D'_i(\theta_j)) = d'_i(\theta_j) = \delta_{ij}$ in k . In other words

$$D'_i(\theta_j) \equiv \delta_{ij} \pmod{m(A)}.$$

Since $D'_i(A^G) \subset A^G$ and since $m(A) \cap A^G = J^G$

$$D'_i(\theta_j) = \delta_{ij} + Q_{ij}(\theta)$$

for certain super polynomials Q_{ij} in the variables θ_i , whose minimal nonvanishing Taylor coefficient has degree ≥ 1 . Suppose $P \neq 0$ is an element in $I = \text{Kern}(f)$ with minimal nonvanishing Taylor coefficient say of degree d , such that this d is minimal among all $0 \neq P \in I$. If $d = 0$, then P is a unit in the superpolynomial ring and the quotient A^G would be zero in contradiction to $1_A \in A^G$. Hence $d > 0$. Let θ_i be a variable which occurs nontrivially in the Taylor coefficient of P of degree d . Then apply the derivative $D'_i(P)$. Obviously $D'_i(P)$ has a nonvanishing Taylor coefficient of degree $d-1$. On the other hand $D'_i(I) \subset I$, hence $D'_i(P) \in I$. This gives a contradiction unless the kernel vanishes $I = 0$. QED

Then by an obvious counting argument lemma 8 implies

Corollary 5. $d_n = \#T_n = \binom{s}{n}$ for all n .

Choice of bases. Up to a scalar $\eta = \theta_I$ for $I = \{1, \dots, s\}$ is independent of the choice of the basis θ_i , since it is a generator of the one dimensional k vectorspace $(J^s)^G$. Hence η is an eigenvalue of the right-invariant operators $D' \in \text{Lie}(G)$ corresponding to the character $\det(J/J^2) = \det(g_-)^{-1}$ of G . η generates A^G as a U -right module for the universal enveloping algebra $U = U(\text{Lie}(A))$.

For the odd superderivations D'_i dual to the $\tilde{\theta}_i \in g_-$ for $i = 1, \dots, s$ define

$$\kappa_A = D'_s \circ \dots \circ D'_1(\eta) \in A^G.$$

Since $Lie(G)$ acts on $k \cdot D'_n \circ \cdots \circ D'_1$ by the character $det(g_-)$, it is easy to see that κ_A is annihilated by all right-invariant derivations D'_X for X in $Lie(G)$. Furthermore $\kappa_A = 1$ modulo $A^G \cap J = J^G$ or

$$e_A^*(\kappa_A) = 1 .$$

A global splitting

The even derivation $D = D_\theta : A \rightarrow A$ defined by the Euler operator

$$D(x) = \sum_{i=1}^s \theta_i \cdot D'_i(x)$$

obviously satisfies $D(A) \subset J$ (with notations as in the last section). Hence as a derivation $D(J^\nu) \subset J^\nu$ for all $\nu \geq 1$. The map

$$E_\nu : J^\nu / J^{\nu+1} \rightarrow J^\nu / J^{\nu+1}$$

induced by D is B -linear. So it suffices to compute E_ν on the basis elements $\tilde{\theta}_I$. [For $x \in J^\nu$ and $a \in A$ use that $D(xa) = xD(a) + D(x)a = D(x)a \bmod J^{\nu+1}$ and $D(A) \in J$ implies $D(xa) = D(x)a \bmod J^{\nu+1}$.] Therefore, as an immediate consequence of $D(\theta_j) = \theta_j$ modulo $J^2 \cap A^G$, this shows

$$E_\nu(\theta_I) = \nu \cdot \theta_I \quad , \quad E_\nu = \nu \cdot id_{J^\nu / J^{\nu+1}} .$$

Lemma 9. *For $char(k) = 0$ or $char(k) > s$ the even derivation $D : A \rightarrow J$ induces an k -linear isomorphism*

$$D : J \cong J .$$

Proof. For large enough ν we have $J^{\nu+1} = 0$. The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J^{\nu+1} & \longrightarrow & J^\nu & \longrightarrow & J^\nu / J^{\nu+1} \longrightarrow 0 \\ & & \downarrow D & & \downarrow D & & \downarrow \nu \cdot id \\ 0 & \longrightarrow & J^{\nu+1} & \longrightarrow & J^\nu & \longrightarrow & J^\nu / J^{\nu+1} \longrightarrow 0 \end{array}$$

commutes. Hence by downward induction $D : J^\nu \rightarrow J^\nu$ is an k -linear isomorphism for all $\nu \geq 1$ using the snake lemma. QED

The kernel

$$\tilde{B} = \text{kernel}(D : A \rightarrow A)$$

of the derivation D is a k -subalgebra of A . In the situation of the last lemma the snake lemma for

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & B \longrightarrow 0 \\ & & \cong \downarrow D & & \downarrow D & & \downarrow 0 \\ 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & B \longrightarrow 0 \end{array}$$

implies that the restriction of the quotient homomorphism $\pi : A \rightarrow B$ to $\tilde{B} \subset A$ is bijective. This inverse of the isomorphism $\pi : \tilde{B} \cong B$ then defines a splitting of $\pi : A \rightarrow B$. Hence we get

Splitting theorem. *Suppose $\text{char}(k) = 0$ or $\text{char}(k) > s$. Then $\pi : \tilde{B} \cong B$ is even and there exists an isomorphism of k -superalgebras*

$$A = A^G \otimes^\varepsilon \tilde{B} \cong k[\theta_1, \dots, \theta_s] \otimes^\varepsilon \tilde{B}.$$

Supergroups

An affine algebraic group G acts on its Lie algebra g_+ by the adjoint representation. Let g_- be any finite dimensional algebraic representation of G over k with action denoted by Ad_- . Then g_+ acts on g_- by derivations $ad_- = \text{Lie}(Ad_-)$. Consider G -equivariant quadratic maps

$$Q : g_- \rightarrow g_+$$

with respect to these actions of G (i.e. arising from a symmetric k -bilinear form on g_- with values in g_+). A triple $\mathbf{G} = (G, g_-, Q)$ as above will be called a supergroup (over k) provided

$$ad_-(Q(v))v = 0$$

holds for all v in g_- . An associated Lie algebra $\text{Lie}(\mathbf{G})$ considered as a \mathbf{Z}_2 -graded Lie algebra structure is defined on $g_+ \oplus g_-$ in the obvious way by the Lie bracket induced by the group structure of G , the action of G on g_- and the map Q (super commutator). See [DM], p.59.

Example 1. If $\mathbf{G} = (G, g_-, Q)$ is a supergroup, then also its connected component in the Zariski topology $\mathbf{G}^0 = (G^0, g_-, Q)$.

Example 2. Super-affine Hopf algebra A define supergroups

$$(Spec(A/J), g_-, Q) ,$$

where Q is the restriction of the Lie bracket on g_- to the diagonal.

Example 3. As a special of example 2 for a finite dimensional super vector space $V = V_+ \oplus V_-$ over k the standard supergroup $Gl(V)$ is defined by $G = Gl(V_+) \times Gl(V_-)$ together with $g_- = Hom_k(V_+, V_-) \oplus Hom_k(V_-, V_+)$ and $Q(A \oplus B) = \{A, B\}$ for the super commutator $\{A, B\} = A \circ B + B \circ A$. Here we used the obvious identification $Lie(Gl(V_{\pm})) = End_k(V_{\pm})$.

Center. For a super group $\mathbf{G} = (G, g_-, Q)$ let the center $Z(\mathbf{G})$ be the maximal central subgroup of G , which acts trivial on g_- .

Morphisms. A homomorphism $(G, g_-, Q) \rightarrow (G', g'_-, Q')$ between supergroups is a pair $\Phi = (\phi, \varphi)$, where $\phi : G \rightarrow G'$ is a group homomorphism between algebraic groups over k and where $\varphi : g_- \rightarrow g'_-$ is a k -linear ϕ -equivariant map such that $Q'(\varphi(X)) = Lie(\phi)(Q(X))$.

Representations. A representation (V, Φ) of a supergroup $\mathbf{G} = (G, g_-, Q)$ is a finite dimensional k super vector space V together with a homomorphism of supergroups $\Phi : (G, g_-, Q) \rightarrow Gl(V)$. The category of such representations, also denoted \mathbf{G} -modules, is a k -linear abelian rigid (monoidal) tensor category

$$Rep_k(\mathbf{G})$$

with the forget functor $(V, \Phi) \mapsto V$ as a super fibre functor. This fiber functor factorizes over the functor

$$Lie : Rep_k(\mathbf{G}) \rightarrow Rep_k(Lie(\mathbf{G})) .$$

The category $Rep_k(Lie(\mathbf{G}))$ of super representations of the Lie superalgebra $Lie(\mathbf{G})$ again is a k -linear abelian rigid (monoidal) tensor category. Notation: Let σ be an automorphism of the supergroup \mathbf{G} . If (V, Φ) is a \mathbf{G} -module, then also $(V, \Phi \circ \sigma)$.

An equivalence of representation categories

Suppose $k = \mathbb{C}$. Let \mathcal{H} be the opposite of the category of affine super Hopf algebras over k . Let \mathcal{HC} be the category of supergroups $\mathbf{G} = (G, g_-, Q)$. Recall G is an affine algebraic groups over k , and morphisms in \mathcal{HC} are algebraic with respect to the first component of the triples. There is an obvious forget functor

$$\mathcal{H} \rightarrow \mathcal{HC} .$$

There is a similar forget functor from the category \mathcal{H}_∞ of differentiable Lie supergroups (as in [DM]) to the category \mathcal{HC}_∞ of differentiable Harish Chandra triples. Objects now are $\mathbf{G}_\infty = (G_\infty, g_-, Q)$ for classical Lie groups G_∞ . According to [DM] p.79, [CF], [K] p. 232 this forget functor is a quasi-equivalence of categories in the C^∞ -case. Consider the following commutative diagram of forget functors

$$\begin{array}{ccc} \mathcal{HC} & \longrightarrow & \mathcal{HC}_\infty \\ \uparrow & & \uparrow \sim \\ \mathcal{H} & \longrightarrow & \mathcal{H}_\infty \end{array}$$

Since an algebraic morphism is determined by its associated C^∞ map, the functor $\mathcal{H} \rightarrow \mathcal{HC}$ is faithful by going over the top of the diagram. We now show

Theorem 4. *The functor $\mathcal{H} \rightarrow \mathcal{HC}$ is fully faithful.*

This immediately implies theorem 3 or the equivalent

Corollary 6. *For a super-affine Hopf algebra A over $k = \mathbb{C}$ with its associated supergroup \mathbf{G} there exists a tensor-equivalence of algebraic tensor categories over k*

$$Rep_k(A) \sim Rep_k(\mathbf{G}) .$$

Proof of theorem. For A, A' in \mathcal{H} with associated triples $Y' = (Spec(B'), g'_-, Q')$ and $Y = (Spec(B), g_-, Q)$ in \mathcal{HC} and a morphism

$$\Phi : Y' \rightarrow Y$$

in \mathcal{HC} we have to construct a homomorphism of super Hopf algebras $\Phi^* : A \rightarrow A'$ inducing Φ . By the diagram above the corresponding differentiable morphism Φ_∞ exists in \mathcal{H}_∞ .

By construction Φ_∞ is ‘reduced algebraic’, i.e. the underlying morphism of Lie groups $G'_\infty \rightarrow G_\infty$ is induced from an algebraic morphism $\Phi_{red} : G' \rightarrow G$ between the underlying reduced algebraic groups. Hence it suffices, if reduced algebraic morphisms Φ_∞ of \mathcal{H}_∞ are induced from algebraic scheme morphisms Φ^* , The algebraic scheme morphism then automatically respects the additional structures comultiplication, antipode and augmentation; this is obvious, since by assumption the C^∞ morphism Φ_∞ induced from it has this property.

To construct Φ^* from a reduced algebraic Φ_∞ consider its graph $\Psi_\infty = (id, \Phi_\infty)$

$$(id, \Phi_\infty) : (G_\infty, g_-, Q) \rightarrow (H_\infty, h_-, Q_H) = (G_\infty, g_-, Q) \times (G'_\infty, g'_-, Q') ,$$

which again is reduced algebraic. By projection onto the second factor it suffices to show that Ψ_∞ is algebraic. Thus it is enough to consider reduced algebraic morphisms Ψ_∞ which are closed immersions. This means that the underlying Hopf algebra morphism

$$\Psi_{red}^* : B \otimes^\varepsilon B' \longrightarrow B$$

is surjective, and that the map $Lie(H_\infty) \hookrightarrow Lie(G_\infty)$ induced by Ψ_∞ is injective.

Construction of Φ^ .* We may assume that Φ_∞ is a locally algebraic closed immersion. How to find Φ^* ? By the splitting theorem it suffices to find a right vertical ring homomorphism $\varphi : A^G \rightarrow (A')^{G'}$

$$\begin{array}{ccccc} A & \cong & \tilde{B} & \otimes & A^G \\ \downarrow \Phi^* & & \downarrow \Phi_{red}^* & & \downarrow \varphi \\ A' & \cong & \tilde{B}' & \otimes & (A')^{G'} \end{array}$$

such that the morphism of super schemes Φ^* induced on the left extends to the given Φ_∞ in the differentiable category. Such φ of course exists if and only if the pullback Φ_∞^* of superfunctions in the differentiable sense satisfies the algebraicity condition

$$\Phi_\infty^*(A^G) \subset (A')^{G'} .$$

Now use $Lie(H) = Lie(H_\infty)$ and $Lie(G) = Lie(G_\infty)$, being defined by left-invariant derivations D_X on the super ring of algebraic resp. differentiable functions. For $X \in g'_+ \subset g_+$ there is a commutative diagram

$$\begin{array}{ccc} C^\infty(Y_\infty) & \xrightarrow{D_X} & C^\infty(Y_\infty) \\ \Phi_\infty^* \downarrow & & \downarrow \Phi_\infty^* \\ C^\infty(Y'_\infty) & \xrightarrow{D_X} & C^\infty(Y'_\infty) \end{array}$$

Since $Lie(G'_\infty) \hookrightarrow Lie(G_\infty)$ the kernel $C^\infty(Y)^G$ of all D_X , $X \in g_+$ derivations on $C^\infty(Y)$ (being contained in the kernel of all D_X for $X \in g'_+$) pulls back to the kernel $C^\infty(Y')^{G'}$ of all $D_{X'}$, $X' \in g'_+$ on $C^\infty(Y')$. Thus the desired existence of φ is evident, if the natural injection

$$A'^{G'} \hookrightarrow C^\infty(Y')^{G'}$$

is a bijection. Notice g'_+ is a Lie algebra, hence integrable! Thus $\dim_k(C^\infty(Y')^{G'}) = 2^s$ for $s = \dim_k(g_-)$ as a consequence of the Frobenius theorem. See [DM], p.75 and [K], p. 230. Therefore

$$\dim_k(A'^{G'}) = 2^s = \dim_k(C^\infty(Y')^{G'}) .$$

This implies $A'^{G'} = C^\infty(Y')^{G'}$ and proves the claim. QED

Semisimple tensor categories

For a k -linear abelian rigid (monoidal) tensor category T with unit object 1_T and $End_T(1_T) = k$ the object 1_T is simple (see [DMi], prop 1.17). Furthermore

Lemma 10. *T is semisimple iff 1_T is injective or projective or $Hom_T(1_T, -)$ is exact or $Ext_T^1(L, 1_T) = 0$ holds for all simple objects L in T .*

Proof. T is semisimple iff $Hom_T(N, M) = Hom_T(1_T, N^* \otimes M) = Hom_T(N \otimes M^*, 1_T)$ is exact in N, M . This is equivalent to $Ext_T^1(L, 1_T) = 0$ for all (simple) objects L in T . QED

For tensor categories T and T' as above let $R : T \rightarrow T'$ be an exact covariant functor with an isomorphism $\iota : 1_{T'} \cong R(1_T)$. Assume $I : T' \rightarrow T$ is a left-exact covariant functor. Let p be an epimorphism in T

$$p : I(1_{T'}) \rightarrow 1_T .$$

Suppose there exists a natural transformation

$$\nu : id \rightarrow R \circ I$$

such that $R(p) \circ \nu_{1_{T'}} = \iota$.

Example. R exact tensor functor with left adjoint I . Then $id \in Hom_T(I(W), I(W))$ defines $\nu_W \in Hom_{T'}(W, RI(W))$ and let $p \in Hom_T(I(1_{T'}), 1_T)$ correspond to $\iota \in Hom_{T'}(1_{T'}, R(1_T))$ for $\iota : 1_{T'} \cong R(1_T)$. Then the above properties hold.

Lemma 11. *a) In the situation above T' is semisimple, if T is semisimple. b) If I is adjoint to R and $End_{T'}(1_{T'}) = k$, then T is semisimple iff T' is semisimple and p splits in T .*

Proof. b) Suppose T' is semisimple. Then $Hom_T(I(1_{T'}), -) = Hom_{T'}(1_{T'}, R(-))$ is exact. If p splits in T , then $1_T \oplus I^+ = I(1_{T'})$. Hence also $Hom_T(1_T, -)$ is exact. Hence T is semisimple. Conversely if T is semisimple, p splits.

a) Suppose T is semisimple. If T' is not semisimple, then by the lemma 10 there exists a simple object L and a nonsplit extension E in T'

$$0 \rightarrow 1_{T'} \xrightarrow{a} E \xrightarrow{b} L \rightarrow 0 .$$

Since $\nu_{1_{T'}} : 1_{T'} \hookrightarrow RI(1_{T'})$ and $RI(a) : RI(1_{T'}) \hookrightarrow RI(E)$ by our assumptions, $a(1_{T'}) \subset E$ is not in the kernel of $\nu_E : E \rightarrow RI(E)$. Hence $b : kern(\nu_E) \rightarrow L$ is a monomorphism. Then $kern(\nu_E) \neq 0$ implies $kern(\nu_E) \cong L$, since L is simple. Since this would split E this proves

$$\nu_E : E \hookrightarrow RI(E) .$$

Since T is semisimple, $I(a) : I(1_{T'}) \hookrightarrow I(E)$ has a section $s : I(E) \rightarrow I(1_{T'})$. Then

$$c : 1_{T'} \rightarrow R(1_T)$$

defined by $c = R(p) \circ R(s) \circ \nu_E \circ a$ is nonzero. [Otherwise $R(s) \circ R(I(a)) = id$, from $s \circ I(a) = id$, would give $\iota = R(p) \circ \nu_{1_{T'}} = R(p) \circ R(s) \circ R(I(a)) \circ \nu_{1_{T'}} = R(p) \circ R(s) \circ \nu_E \circ a = 0$ by the naturality $\nu_E \circ a = RI(a) \circ \nu_{1_{T'}}$ of ν]. Hence c is an isomorphism as $R(1_T) \cong 1_{T'}$ is simple, using $End_{T'}(1_{T'}) = k$. Then $c^{-1} \circ R(p \circ s)$ splits $\nu_E(E)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & kernel & \longrightarrow & \nu_E(E) & \xrightarrow{R(p \circ s)} & R(1_T) \longrightarrow 0 \\ & & & & \uparrow & \nearrow \cong & \\ & & & & \nu_E(a(1_{T'})) & \nearrow c & \end{array}$$

Since $\nu_E(E) \cong E$ this splits $a(1_{T'})$ in E . Contradiction! Hence T' is semisimple.

Semisimple representation categories

Let $\mathbf{G} = (G, g_-, Q)$ be a supergroup over $k = \mathbb{C}$. The obvious covariant exact restriction functor $R : \text{Rep}_k(\mathbf{G}) \rightarrow \text{Rep}_k(G)$ satisfies $R(k) \cong k$. There exists a covariant induction functor

$$I : \text{Rep}_k(G) \rightarrow \text{Rep}_k(\mathbf{G})$$

which, for V in $\text{Rep}_k(G)$ and $g = \text{Lie}(\mathbf{G})$, is defined by

$$I(V) = U(g) \otimes_{U(g_+)}^\varepsilon V.$$

The action of $g_+ = \text{Lie}(G)$ on $I(V)$ comes from an algebraic action of G on $I(V)$ by the g_+ -module isomorphism $I(V) \cong \Lambda^\bullet(g_-) \otimes^\varepsilon V$. Hence $I(V) \in \text{Rep}_k(\mathbf{G})$. It is easy to see that I is exact and left adjoint to R , i.e. Frobenius reciprocity $\text{Hom}_{\mathbf{G}}(I(V), W) = \text{Hom}_G(V, R(W))$.

Since k has characteristic zero $\text{Rep}_k(G)$ is semisimple if and only if G is a reductive algebraic group over k . Therefore lemma 11 b) implies

Theorem 5. *$\text{Rep}_k(\mathbf{G})$ is semisimple if and only if (a) G is reductive and (b) the surjection of \mathbf{G} -modules defined by the adjunction morphism*

$$ad : I(k) \rightarrow k$$

has a splitting in the category $\text{Rep}_k(\mathbf{G})$.

Remark. By $\text{char}(k) = 0$ condition a) holds iff $g_+ = \text{Lie}(G)$ is a reductive Lie algebra over k . Condition b) says that the restriction

$$ad : I(k)^{\mathbf{G}} \rightarrow k$$

to the space of \mathbf{G} -invariant subspace of $I(k)$ is surjective. For $g = \text{Lie}(\mathbf{G})$ then $I(k)^{\mathbf{G}} = (I(k)^g)^G = (I(k)^g)^{\pi_0(G)}$ by [DG], prop. 2.1(c), p.309. The group of connected components $\pi_0(G)$ of G in the Zariski topology is finite. Since $\text{char}(k) = 0$ the functor of $\pi_0(G)$ -invariants is exact by Maschke's theorem. Hence conditions a) resp b) are equivalent to the following conditions

- a') The Lie algebra g_+ is reductive.
- b') The restriction of $ad : I(k) \rightarrow k$ to $I(k)^g$ is surjective.

Definition. If g satisfies these two properties, we say the Lie superalgebra g is *reductive*. $Rep_k(\mathbf{G})$ is semisimple if and only if $g = Lie(\mathbf{G})$ is reductive (by theorem 5). In this case we say \mathbf{G} is reductive.

Connected component. As already explained

1. \mathbf{G} is reductive if and only if its connected component \mathbf{G}^0 is reductive.

Etale coverings. Similarly we may replace G by a finite etale covering $G' \rightarrow G$. We say that the supergroup $\mathbf{G}' = (G', g_-, Q)$ attached to $\mathbf{G} = (G, g_-, Q)$ is a finite etale (central) cover of \mathbf{G} . Then of course

2. \mathbf{G} is reductive if and only if the etale cover \mathbf{G}' is reductive.

Then $\mathbf{G} = \mathbf{G}'/F$ for a finite subgroup F of the center $Z(\mathbf{G}')$ of \mathbf{G}' .

Central quotients. Finally if Z is a closed subgroup of the center $Z(\mathbf{G})$ of \mathbf{G} , then $\mathbf{G}/Z = (G/Z, g_-, Q)$ again a supergroup called a central quotient. Obviously

3. If \mathbf{G} is reductive, then any central quotient \mathbf{G}/Z is also reductive

since $Rep_k(\mathbf{G}/Z)$ is a full subcategory of $Rep_k(\mathbf{G})$, if $Rep_k(\mathbf{G})$ is semisimple!

Reductive supergroups

The main classification statement involves the orthosymplectic supergroups

$$Spo(1, 2r) = (Sp(2r, J), k^{2r}, Q) .$$

Fix a nondegenerate antisymmetric $2r \times 2r$ -matrix $J' = -J$ so that $g \in Sp(2r, k) \Leftrightarrow g'Jg = J$. This identifies $sp(2r, J)$ with the matrices X for which JX is symmetric. For the standard action Ad_- of $Sp(2r, J)$ on k^{2r} the map $Q : k^{2r} \rightarrow sp(2r, J)$

$$Q(v)_{\alpha\beta} = \sum_{\gamma=1}^{2r} v_{\alpha} v_{\gamma} J_{\gamma\beta}$$

for $v = (v_1, \dots, v_{2r}) \in k^{2r}$ is well defined and equivariant such that $Q(v)v = 0$. So this defines a supergroup. Different choices of J yield isomorphic supergroups.

Proposition 1. *A supergroup is reductive over $k = \mathbb{C}$ if and only if its connected component admits a finite etale central covering, which as a supergroup is a direct product of super groups of the following type*

1. *A classical central k -torus*
2. *Simple connected simply connected classical k -groups*
3. *Simple supergroups of orthosymplectic type $Spo(1, 2r)$ for integers $r \geq 1$.*

Similarly a Lie superalgebra is reductive if and only if, modulo the center, it is a direct sum of simple Lie superalgebras of classical type or of the orthosymplectic types $BC_r = spo(1, 2r)$ corresponding to the super groups $Spo(1, 2r)$.

Proof. A product of reductive supergroups is reductive. We leave this as an exercise. So in one direction it suffices that the supergroups $Spo(1, 2r)$ are reductive. In fact $Rep_k(Spo(1, 2r)) = Rep_k(spo(1, 2r))$ because $Sp(2r, J)$ is simply connected, and this reduces to [DH], theorem 4.1.

Now for the converse. By our preliminary remarks in the last section we may replace \mathbf{G}^0 by an etale finite covering \mathbf{G}' , where $G' = T \times S$ for a k -torus T and a product S of connected simple and simply connected k -groups. Then we can divide \mathbf{G}' by its maximal central torus Z . The new supergroup \mathbf{G}'' is reductive, if \mathbf{G} is reductive. This allows to reduce the proof to the case $\mathbf{G} = (G, g_-, Q)$ without central torus so that in addition G is connected and a product of a torus T and a simple simply connected k -group S . If these conditions hold and $Rep_k(\mathbf{G})$ is semisimple, we say \mathbf{G} is *good*. So assume \mathbf{G} is good. Then by [DG], page 309ff and theorem 4 it suffices to prove that $g = Lie(\mathbf{G})$ is a product of simple Lie superalgebras of the classical type and types BC_r . Using condition b') this immediately would follow from [DH], theorem 4.1 for semisimple g_+ .

We already know g_+ is reductive. To show that g_+ is semisimple we claim that g is a direct sum of Lie superalgebras g_ν with $(g_\nu)_+ \neq 0$ and either $(g_\nu)_- = 0$ or $(g_\nu)_-$ is an irreducible $(g_\nu)_+$ -module with $(g_\nu)_+ = [(g_\nu)_-, (g_\nu)_-]$. This is easy: For $g_- = s \oplus t$ and an irreducible g_+ -submodule s

$$h_+ = [s, s]$$

is an ideal in g_+ by the Jacobi identity $[g_+, [s, s]] \subset [s, [g_+, s]] + [[g_+, s], s] \subset [s, s]$. Hence either $h = h_+$ in case h_+ commutes with g_- , or otherwise

$$h = h_+ \oplus s,$$

is an ideal in g with the desired property. (As a \mathbf{G} -module, thus as a g -module) $g = h \oplus h'$ splits into ideals by the semisimplicity of $Rep_k(\mathbf{G})$. The ideal property

$[h, h'] \subset h \cap h' = 0$ decomposes g . Since condition b') easily implies $h_+ \neq 0$ for $s \neq 0$ (see [DH], prop.2.2) our claim follows by induction.

To show that $h = h_+ \oplus s$ is an ideal for $[h_+, g_-] \neq 0$, notice $[h_+, t] = 0$. Indeed $[h_+, t] \subset [g_+, t] \subset t$ and $[h_+, t] = [[s, s], t] \subset [[s, t], s] \subset [g_+, s] \subset s$. Thus $[h_+, s] = [h_+, g_-] \neq 0$. Therefore $s = [h_+, s]$, since $[h_+, s]$ is a g_+ -submodule of s . Obvious are $[g_+, s] \subset s$ and $[g_+, h_+] \subset h_+$ and similarly $[g_-, h_+] = [g_-, [s, s]] \subset [[g_-, s], s] \subset [g_+, s] \subset s$. To show $[g_-, s] = [t, s] + [s, s] \subset h_+$ use $[t, s] = [t, [h_+, s]] \subset [[h_+, t], s] + [[t, s], h_+] = [[t, s], h_+] \subset [g_+, h_+] \subset h_+$.

If $(g_\nu)_- \neq 0$ is an irreducible $(g_\nu)_+$ -module, the center z_ν of $(g_\nu)_+$ acts by a character χ_ν . By the equivariance and surjectivity (!) of the Lie bracket

$$(g_\nu)_- \times (g_\nu)_- \rightarrow (g_\nu)_+ \neq 0$$

the trivial action of z_ν on $(g_\nu)_+$ forces $2\chi_\nu = 0$, hence $\chi_\nu = 0$. Thus z_ν is in the center of g , therefore trivial by our assumption that \mathbf{G} is good. Hence the reductive Lie algebra g_+ is semisimple. QED

The categories $Rep_k(\mathbf{G}, \varepsilon)$

For a supergroup $\mathbf{G} = (G, g_-, Q)$ suppose ε is in the center of $G(k)$ such that $\varepsilon^2 = 1$ and $Ad_-(\varepsilon) = -id_{g_-}$. Let $T = Rep_k(\mathbf{G}, \varepsilon)$ be the full subcategory of $Rep_k(\mathbf{G})$ defined by the super representations (V, ϕ, φ) for which $\phi(\varepsilon) = \sigma_V$ is the super parity automorphism σ_V of V . T is an algebraic tensor category over k (see [D]).

Not every supergroup $\mathbf{G} = (G, g_-, Q)$ admits twisting elements ε as above. But the extended supergroup $\mathbf{G}^{ext} = (G \times \mu_2, g_-, Q)$, where $Ad_-(g, \pm 1) = \pm Ad_-(g)$, always has the twisting element $\varepsilon^{ext} = (1, -1) \in G^{ext} = G \times \mu_2$. The forget functor defines a tensor-equivalence

$$Rep_k(\mathbf{G}^{ext}, \varepsilon^{ext}) = Rep_k(\mathbf{G}),$$

since $(V, \phi, \varphi) \in Rep_k(\mathbf{G})$ extends uniquely to $(V, \phi^{ext}, \varphi) \in Rep_k(\mathbf{G}^{ext}, \varepsilon^{ext})$ for $\phi^{ext}(g, \pm 1) = \sigma_V \phi(g) = \phi(g) \sigma_V$.

Lemma 12. *$Rep_k(\mathbf{G}, \varepsilon)$ is semisimple if and only if $Rep_k(\mathbf{G})$ is semisimple.*

Since this is a statement on the underlying abelian categories, we may ignore the tensor structures on these categories. On the underlying abelian categories the parity change $\Pi(V) = V \otimes^\varepsilon \bar{1}$, defined by the trivial super representation $\bar{1} = \Pi(1)$ on $k^{0|1}$, induces a functor $\Pi : \text{Rep}_k(\mathbf{G}) \rightarrow \text{Rep}_k(\mathbf{G})$ which in general does not preserve the subcategory $\text{Rep}_k(\mathbf{G}, \varepsilon)$. However

$$\Pi : \text{Rep}_k(\mathbf{G}^{ext}) \rightarrow \text{Rep}_k(\mathbf{G}^{ext})$$

preserves the subcategory $\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext})$.

Proof of lemma 12. In the extended supergroup \mathbf{G}^{ext} we have two twisting elements ε and ε^{ext} . This defines an element $z = \varepsilon \varepsilon^{ext} = (\varepsilon, -1) \in G \times \mu_2$ in the center of the supergroup \mathbf{G}^{ext} , i.e. z is in the center of G^{ext} with trivial action on g_- , and z commutes with ε and ε^{ext} . The eigenspace decomposition with respect to z decomposes the category

$$\text{Rep}_k(\mathbf{G}^{ext}) = \text{Rep}_k^+(\mathbf{G}^{ext}) \oplus \text{Rep}_k^-(\mathbf{G}^{ext})$$

and also its subcategories $\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext})$ and $\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon)$. Then by definition $\text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon^{ext}) = \text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon)$ and $\text{Rep}_k^-(\mathbf{G}^{ext}, \varepsilon^{ext}) = \Pi(\text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon^{ext}))$, since ε has trivial action and ε^{ext} acts by -1 on $\bar{1} \in \text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext})$. Ignoring tensor structures $T = \text{Rep}_k(\mathbf{G}, \varepsilon) = \text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon) = \text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon^{ext})$ and $\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext}) = \text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon^{ext}) \oplus \Pi(\text{Rep}_k^+(\mathbf{G}^{ext}, \varepsilon^{ext}))$ give

$$\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext}) = T \bigoplus \Pi(T) \quad , \quad T = \text{Rep}_k(\mathbf{G}, \varepsilon) .$$

Hence T is semisimple iff $\text{Rep}_k(\mathbf{G}^{ext}, \varepsilon^{ext}) = \text{Rep}_k(\mathbf{G})$ is semisimple. QED

Remarks on $\mathbf{G} = \text{Spo}(1, 2r)$

We discuss the representations of the orthosymplectic group over $k = \mathbb{C}$. The category $\text{Rep}_k(\mathbf{G})$ of super representations of a supergroup \mathbf{G} contains the trivial even representation 1 on $k = k^{1|0}$ and the odd trivial representation $\bar{1}$ on $k^{0|1}$ such that $\bar{1} \otimes^\varepsilon \bar{1} = 1$.

For $\mathbf{G} = \text{Spo}(1, 2r)$ the center of $G = \text{Sp}(2r, J)$ is μ_2 . The center of \mathbf{G} is trivial. Hence $\varepsilon = -id$ gives a unique choice for a twisting element ε to define a category $T = \text{Rep}_k(\mathbf{G}, \varepsilon) \subset \text{Rep}_k(\mathbf{G})$. Recall from the last section

$$\text{Rep}_k(\mathbf{G}) = T \bigoplus \Pi(T) .$$

Also notice $\Pi(W) = W \otimes^\varepsilon \bar{\mathbb{I}}$.

The standard representation V . Consider the following representation $(V, \phi, \varphi) \in \text{Rep}_k(\mathbf{G}, \varepsilon)$ of the supergroup on $\mathbf{G} = \text{Spo}(1, 2r)$. As a G -module $V = V_+ \oplus V_- = k \oplus g_-$ with trivial action on $V_+ = k$ and with the standard representation of G on V_- . This defines $\phi(X) \in \text{End}(V)_+$ for $X \in g_+$. We identify V_- with g_- . The odd elements $v \in g_-$ act on V by $\varphi(v) \in \text{End}(V)_-$ defined by the annihilation and creation operators

$$\varphi(v)w = \frac{1}{2}v'Jw \in V_+ \quad , \quad w \in V_-$$

$$\varphi(v)\lambda = \lambda \cdot v \in V_- \quad , \quad \lambda \in V_+ .$$

Then $\phi(Q(v)) = [\varphi(v), \varphi(v)]$ for $v \in g_-$. We call V the orthosymplectic standard representation. It is easy to see that V is an irreducible super representation.

Invariant form b . The orthosymplectic standard representation V admits a nondegenerate supersymmetric \mathbf{G} -invariant form

$$b : V \otimes^\varepsilon V \rightarrow k^{1|0}$$

where b is the orthogonal direct sum of the symmetric form $b(\lambda_1, \lambda_2) = \lambda_1\lambda_2$ on $V_+ = k$ and the antisymmetric form $b(v_1, v_2) = -\frac{1}{2}v'_1Jv_2$ on V_- . In fact the orthosymplectic supergroup \mathbf{G} is the automorphism group of this supersymmetric form b on V . In particular: The standard representation V is an ‘orthogonal self dual’ faithful representation of \mathbf{G} . Hence V is a tensor generator of

$$T = \text{Rep}_k(\mathbf{G}, \varepsilon) = \langle V \rangle .$$

See [Sh] for an explicit decomposition of the tensor powers $V^{\otimes r}$. See [RS] for a connection of T with the representation category of the group $SO(2r+1)$.

Lemma 13. *All irreducible representations in T are ‘orthogonal self dual’. All representations in $\Pi(T)$ are ‘symplectic self dual’.*

Proof. If W is ‘orthogonal self dual’ then $\Pi(W)$ is ‘symplectic self dual’ and vice versa. Since $\text{Rep}_k(\mathbf{G}) = T \oplus \Pi(T)$ it therefore suffices that T contains all ‘orthogonal self dual’ irreducible representations. Tensor products of ‘orthogonal self dual’ representations are ‘orthogonal self dual’, hence any multiplicity one representation contain in it is again ‘orthogonal self dual’. By the theory

of highest weight vectors any irreducible representation W in $Rep_k(\mathbf{G})$ appears with multiplicity one in a tensor power of irreducible fundamental representations $V_i, i = 1, \dots, r$ of \mathbf{G} up to parity shift. For these $(\bar{1}^{\otimes i} \otimes^\varepsilon V_i)_+ = \Lambda^i(g_-)$ and $(\bar{1}^{\otimes i} \otimes^\varepsilon V_i)_- = \Lambda^{i-1}(g_-)$. See [Dj], p.31 and p.36. Obviously $V_i \in Rep_k(\mathbf{G}, \varepsilon)$. The V_i are self dual, therefore ‘orthogonal self dual’ by considering their restriction to G , which contains the highest weight representation with multiplicity one as an ‘orthogonal self dual’ representation of G . QED

We claim

Lemma 14. *For $\mathbf{G} = Sp(1, 2r)$ the tensor subcategory of $Rep_k(\mathbf{G})$ generated by the standard representation $V = k^{1|2r}$ of \mathbf{G} is $Rep_k(\mathbf{G}, \varepsilon)$. The tensor subcategory generated by $\Pi(V)$ is the full category $Rep_k(\mathbf{G})$.*

Proof. It suffices to find $\bar{1} = \Pi(1)$ in a tensor power of $\Pi(V)$. Then $V = \Pi(V) \otimes^\varepsilon \bar{1}$ generates T and $T \oplus (T \otimes^\varepsilon \bar{1}) = Rep_k(\mathbf{G})$. We claim

$$\Pi(1) \hookrightarrow \Pi(I(1)) \cong \Lambda^{2r+1}(\Pi(V))$$

for the induced module $I(k) = I(1)$. By Frobenius reciprocity the dimension of

$$End_{\mathbf{G}}(I(k)) \cong Hom_G(k, I(k)) \cong (\Lambda^\bullet(g_-))^G$$

is $r + 1$ by the classical invariant theory of the group $G = Sp(2r)$. A basis for the invariants are the powers ω^i of the symplectic form $\omega \in \Lambda^2(g_-)$. Indeed

$$I(k) = \bigoplus_{i=0}^r V_i \in Rep_k(\mathbf{G}, \varepsilon)$$

for $V_0 = 1$ and the different fundamental representations V_1, \dots, V_r of \mathbf{G} (see [Dj], p.36). By Frobenius reciprocity also the dimension of

$$Hom_{\mathbf{G}}(I(k), \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \bar{1}) = Hom_G(k, \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \bar{1})$$

is equal to $r + 1$ using

$$\Lambda^{2r+1}(\Pi(V))^G = \bigoplus_{j=0}^{2r+1} \Lambda^j(g_-)^G \otimes^\varepsilon \bar{1}^{\otimes(2r+1-j)} \cong \bigoplus_{j=0}^r \bar{1}^{\otimes(2r+1-2j)}.$$

Then $I(k) \cong \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \bar{1}$, provided $\Lambda^{2r+1}(\Pi(V)) = \Pi(\text{Sym}^{2r+1}(V))$ has at least $r+1$ nonisomorphic irreducible constituents. For this (with the convention I of [DM], p.49 and p.62f) consider the superpolynomial ring $S^\varepsilon(V) = \text{Sym}^\bullet(V)$

$$\text{Sym}^\bullet(V) = \text{Sym}^\bullet(V_+) \otimes^\varepsilon \Lambda^\bullet(V_-) .$$

Multiplication with the invariant form $b \in \text{Sym}^2(V)$ is injective inducing a filtration $F_0 \hookrightarrow F_1 \hookrightarrow \dots \hookrightarrow F_r$ of F_r by \mathbf{G} -modules $F_i = \text{Sym}^{2i+1}(V) \otimes^\varepsilon b^{\otimes(r-i)}$. Notice $F_i \cong F_{i+1}$ for $i \geq r$. The highest weight submodules of the $G_i = F_i/F_{i-1}$ for $i = 0, \dots, r$ define $r+1$ nonisomorphic G -modules, since $(G_i)_- \cong \Lambda^{2i+1}(V_-)$ and $(G_i)_+ \cong \Lambda^{2i}(V_-)$ as G -modules. Hence

$$I(1) \cong \text{Sym}^{2r+1}(V) .$$

Therefore the G_i must have been irreducible \mathbf{G} -modules. Considering highest weights a comparison shows $G_i \cong V_{2i+1}$ for $0 \leq i < \frac{r}{2}$ and $G_{r-i} \cong V_{2i}$ for $0 \leq i \leq \frac{r}{2}$. Hence all the representations V_i for $i = 0, \dots, r$ are constituents of the tensor power $V^{\otimes(2r+1)}$. QED

We remark that there is a dual filtration $F'_i = \text{Sym}^{2i}(V) \otimes^\varepsilon b^{\otimes(r-i)}$ on $\text{Sym}^{2r}(V)$ with $G'_i \cong G_{r-i}$, and again $I(1) \cong \text{Sym}^{2r}(V)$.

Lemma 15. *Let T be a semisimple algebraic tensor category over an algebraically closed field k of characteristic zero. For a simple object $W \neq 0$ of T the categorial rank $rk_k(W)$ does not vanish.*

Proof. Since $rk_k(W) = sdim_k(W)$, this follows from [Ka], p.619 formula (2.6) with $B(0, n) = spo(2n, 1)$ in the notations of loc. cit. QED

Structure Theorem

Assume $k = \mathbb{C}$. Then according to proposition 1 a connected reductive supergroup \mathbf{G} is of the form $\mathbf{G} = (\mathbf{G}' \times H)/F$ where $\mathbf{G}' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ is a product of orthosymplectic supergroups and where H is a reductive algebraic k -group. Since F is a finite central subgroup of $\mathbf{G}' \times H$ and since the center of \mathbf{G}' is trivial, this implies $F \subset H$. Hence $\mathbf{G} = \mathbf{G}' \times H'$ for $H' = H/F$. Hence

Lemma 16. *A connected reductive supergroup \mathbf{G} is isomorphic to a product $\mathbf{G}' \times H$ where H is a reductive algebraic k -group and where $\mathbf{G}' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ is a product of orthosymplectic supergroups.*

For $\mathbf{G} = Spo(1, 2r)$ and $G = Sp(2r)$ one has $Aut(G) = G_{ad}$ and therefore $Aut(\mathbf{G}) = G$. In other words, any automorphism of \mathbf{G} is an inner automorphism $Int(g)$ for a unique element $g \in G$. Let \mathbf{G} be a reductive supergroup. Then the group $\pi_0(\mathbf{G}) = \pi_0(G)$ acts on \mathbf{G}' . For $\bar{g} \in \pi_0(G)$ we can choose a representative $g \in G$, by a suitable modification with an element in $G' = \prod_{r \geq 1} Sp(2r)^{n_r}$, such that g acts by a strict permutation of the factors on \mathbf{G}' . The group of such $g \in G$ defines a canonical subgroup $G_1 \subset G$ such that $G_1 \cap G' = 1$. Hence $G_1 \subset H$. Hence any $\bar{g} \in \pi_0(G) = \pi_0(H)$ has a representative in $G_1 \subset H$. We get a canonical homomorphism

$$p : H \rightarrow \prod_{r \geq 1} \Sigma_{n_r}$$

into the product of symmetric permutation groups Σ_{n_r} whose kernel is G_1 . Conversely given such a homomorphism $p : H \rightarrow \prod_{r \geq 1} \Sigma_{n_r}$ for a reductive algebraic k -group H one can construct the semidirect product supergroup $\mathbf{G} = \mathbf{G}' \triangleleft H$ obtained from the permutation action of H on $\mathbf{G}' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$. Obviously in our case therefore

Theorem 6. *Any reductive supergroup \mathbf{G} over an algebraically closed field k of characteristic zero is isomorphic to a semidirect product $\mathbf{G}' \triangleleft H$ of a reductive algebraic k -group H with a product $\mathbf{G}' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ of simple supergroups of BC-type, where the semidirect product is defined by an abstract group homomorphism*

$$p : \pi_0(H) \rightarrow \prod_{r \geq 1} \Sigma_{n_r} .$$

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